The bisection of random graphs: rigorous bounds using the analysis of algorithms.

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Abstract

Given a random graph $G = G_{n,1/2}$, we give almost sure bounds for the value of the minimum bisection of G. The lower bound is obtained by applying the first moment method, while the upper bound is given by the analysis of a simple algorithm. Moreover, these results enable us to derive upper bounds for the maximum bisection and the maxcut of G.

Keywords: random graph, graph bisection, maxcut, first moment method, randomized algorithm, analysis of algorithms.

1 Introduction

A central topic in physics is the investigation of properties of spin glasses. A number of theories have been developed over the years in order to model spin glasses and try to explain their behavior. One of the most commonly used models is the Ising model both for simplicity and its accuracy in representing real world situations (for more details on this topic, the reader may consult [3]).

A spin glass is an alloy of magnetic impurities diluted in a nonmagnetic metal. Alloys that show spin glass behavior are, for instance , CuMn, the metallic crystal AuFe, or the insulator EuSrS. At very low temperature the spin glass system attains a minimum energy configuration, referred as ground state. this state can be found by minimizing the hamiltonian representing the total energy of the system; this problem turns out to be a max-cut problem in certain graphs.

Let us assume that the given spin glass system consist of n magnetic impurities (atoms). Each magnetic atom i has a magnetic orientation (spin) which, in the Ising model, is represented by a variable $s_i \in \{-1, 1\}$ (meaning magnetic north pole 'up' or 'down'). Between any pair i, j of atoms there is an interaction J_{ij}, which depends on the nonmagnetic material and the distance r_{ij} between the atoms. The orientation of the exterior magnetic field is represented by $s_0 \in \{-1, 1\}$ and its strength by a constant h. Then the energy of the system is given by the hamiltonian:

$$H = -\sum_{1 \le i < j \le n} J_{ij} s_i s_j - h \sum_{i=1}^{n} s_0 s_i.$$

Finding the extremum of this function is thus a most important problem for the physicists. It is easily seen to be an instance of a graph partitioning problem. Indeed, if $(J_{ij})_{i,j}$ are the coefficient of the adjacency matrix of the graph and s_i is 1 or -1 depending on which side of the partition the vertex i (s_0 is 0, the exterior has no influence) is placed, then the quantity $\sum_{1 \le i < j \le n} \frac{J_{ij}}{2}(1 - s_i s_j)$ is the size of the cut.

In this paper, we study several types of graph cuts. In the following, G will denote the unweighted, undirected random graph with n vertices and whose every edge exists with probability 1/2 (for further details, see [1]). These cuts are defined as follow:

Minbisection:
Input: a graph $G = (V(G), E(G))$ with an even number of edges,
Output: 2 subsets of equal size of $V(G)$ such that the number of edges with exactly one endpoint in each subset is minimum. This minimum number of edges is denoted by $\mathrm{Bi}(G)$.

Maxbisection:
Input: a graph $G = (V(G), E(G))$ with an even number of edges,
Output: 2 subsets of equal size of $V(G)$ such that the number of edges with exactly one endpoint in each subset is maximum. This maximum number of edges is denoted by $mc_{n/2}(G)$.

Maxcut:
Input: a graph $G = (V(G), E(G))$,
Output: 2 subsets of $V(G)$ such that the number of edges with exactly one endpoint in each subset is maximum. This maximum number of edges is denoted by $mc(G)$.

Using the controversial replica method, the physicists have obtained the following result (see[3]) :

$$\mathrm{Bi}(G) = \frac{n^2}{8} - 0.3816 n^{3/2} + o(n^{3/2}), \tag{1}$$

The following theorems 1 and 3 provide the first almost sure bounds for the minimum bisection of random graph (to be compared with 1). In section 2, the Theorem 1 and its Corollary are proved.

Theorem 1 *For a random graph $G_{n,1/2}$, we have with high probability*

$$Bi(G) \geq \frac{n^2}{8} - \frac{\sqrt{\ln 2}}{2\sqrt{2}} n^{3/2}. \tag{2}$$

Corollary 1 *The maximum bisection of a random graph G verifies almost surely*

$$mc_{n/2}(G) \leq \frac{n^2}{8} + \frac{\sqrt{\ln 2}}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

This Corollary enables us to prove the Theorem 2 in section 3:

Theorem 2 *The maximum bisection of a random graph G verifies almost surely*

$$mc_{n/2}(G) \leq \frac{n^2}{8} + \frac{\sqrt{\ln 2}}{2\sqrt{2}} n^{3/2} + o(n^{3/2}).$$

And finally, in section 4, we present an algorithm named $ALBI$ which, given a graph G, outputs a bisection. An analysis of this algorithm enables us to prove the following Theorem:

Theorem 3 *The minimum bisection $|widetildeBi(G)$ of the random graph $G = G_{n,1/2}$ satisfies almost surely*

$$\widetilde{Bi}(G) \leq \frac{n^2}{8} - 0.133 n^{3/2}.$$

2 Proofs

2.1 Proof of Theorem 1

Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let (X, Y) be a balanced partition of $V(G)$. The number of edges of G between X and Y will be denoted by $e(X, Y)$. We have thus

$$\mathrm{Bi}(G) = \max_{(X,Y)} e(X, Y)$$

where (X, Y) ranges over all balanced bipartition of $V(G)$.
We are now going to prove the theorem 1. The proof uses the first moment method. We have:

$$e(X, Y) = \sum_{\substack{x \in X \\ y \in Y}} 1_{\{xy \in E(G)\}}.$$

We can now compute the mathematical expectation of $e(X, Y)$, for a fixed bipartition (X, Y):

$$\mathbb{E}[e(X, Y)] = \mathbb{E}[\sum_{\substack{x \in X \\ y \in Y}} 1_{\{xy \in E(G)\}}] = \sum_{\substack{x \in X \\ y \in Y}} \mathbb{E}[1_{\{xy \in E(G)\}}] = \frac{n^2}{8}.$$

It follows that there exists a least 1 partition (X,Y) such that $e(X,Y) \leq n^2/8$. Our goal is to found the order of magnitude of $n^2/8 - \mathrm{Bi}(G)$.

In our model of random graphs, it is quite clear that $e(X,Y)$ follows a probability law $\mathrm{B}(\frac{n^2}{4}, \frac{1}{2})$ where $\mathrm{B}(m,p)$ denotes the binomial distribution with parameters m and p. We are now going to show that, with high probability, the size of the minbisection is more than $\frac{n^2}{8} - \frac{\lambda n}{4}$, for a λ that we will compute. The proof uses the first moment method. We compute λ such that $\mathbb{E}[\#(X,Y) : e(X,Y) \geq \frac{n^2}{8} - \frac{\lambda n}{4}]$ is $o(1)$. Fix $\lambda \in \mathbb{R}^+$.

$$\mathbb{E}[\#(X,Y) : e(X,Y) \leq \frac{n^2}{8} - \frac{\lambda n}{4}] \;=\; \sum_{(X,Y)} \mathbb{P}\mathrm{r}[e(X,Y) \leq \frac{n^2}{8} - \frac{\lambda n}{4}],$$

where the sum is taken on all the possible bisections (X,Y).

Using the normal approximation for the law of $e(X,Y)$, we have

$$\begin{aligned}
\mathbb{P}\mathrm{r}[e(X,Y) \leq \frac{n^2}{8} - \frac{\lambda n}{4}] &\sim \mathbb{P}\mathrm{r}[\mathcal{N}(\frac{n^2}{8}, \frac{n^2}{16}) \leq \frac{n^2}{8} - \frac{\lambda n}{4}] \\
&\sim \mathbb{P}\mathrm{r}[\mathcal{N}(0,1) \leq -\lambda] \\
&\leq e^{-\frac{\lambda^2}{2}}.
\end{aligned}$$

Since $\#(X,Y) \leq 2^n$, and we obtain

$$\mathbb{E}[\#(X,Y) : e(X,Y) \leq \frac{n^2}{8} + \frac{\lambda n}{4}] \;\leq\; e^{-\frac{\lambda^2}{2} + n \ln 2}.$$

The right-hand side is $o(1)$ for

$$\lambda = -\sqrt{2 \ln 2}\sqrt{n}.$$

The Theorem 1 follows immediately from this statement and (3).

2.2 Proof of Corollary 1

It is clear that if (X,Y) defines a minimum bisection of G, then (X,Y) defines also a maximum bisection of the complementary graph of G. The distribution of G remains unchanged by complementarity, so we can deduce the following theorem from the previous section.

3 Proof of Theorem 2

The proof of Theorem 2 consists in proving that a partition realizing the maxcut is almost balanced and that from such a partition we can deduce a bipartition whose number of edges is close from the maximum. Formely we prove the following assertions:

(i) the unbalance of a partition giving a maximum cut is almost surely smaller than $o(n^{3/4})$.

(ii) from every unbalanced cut (X, Y), such that $||X| - |Y|| = \delta \leq n^{3/4}$, we can deduce a balanced cut of the same graph with a loss smaller than $o(n^{3/2})$.

Proof of (i) Let $G = G_{n,1/2}$, and let (X, Y) be a partition of the vertex set such that $|X| \geq n/2 + f(n)$ and $|Y| \leq n/2 - f(n)$, for a function f. We show that if $f \notin o(n^{3/4})$ the cut is not maximum with high probability. Let us put $\delta = f(n)$. The variable $e(X, Y)$ follows a binomial distribution $\mathcal{B}(m, p)$ with $m = n^2/4 - f^2(n)$ and $p = 1/2$, so the Chernoff bound gives

$$\mathbb{P}\mathrm{r}\left[e(X, Y) > \frac{n^2}{8}\right] \leq exp\left(-\frac{f^4(n)}{n^2}\right).$$

The expectancy of the number of partitions (X, Y) defined as above such that $e(X, Y) > n^2/8$ is then

$$\begin{aligned} \mathbb{E}\left[\#(X, Y) : e(X, Y) > \frac{n^2}{8}\right] &= 2^n \mathbb{P}\mathrm{r}\left[e(X, Y) > \frac{n^2}{8}\right] \\ &\leq \exp\left(n \ln 2 - \frac{f^4(n)}{n^2}\right). \end{aligned}$$

This expectation tends to 0 if $f(n) \geq cn^{3/4}$ (with $c > (\ln 2)^{1/4}$ Since a maximum cut is greater than $n^2/8$, we have with high probability that $f(n) \leq cn^{3/4}$.

*Proof of (ii)*We consider again a random graph as previously, and suppose that the maxcut is obtained by a partition (X, Y) with $|X| = n/2 + \delta$ and $|Y| = n/2 - \delta$, $\delta \leq cn^{3/4}$ for a constant c. Let Z be a randomly chosen subset of X, such that $|Z| = \delta$. We set $X' = X - Z$ and $Y' = Y + Z$ so that (X', Y') is balanced. We aim to prove the following statement via a contradiction.

$$e(X', Y') - e(X, Y) = o(n^{3/2}).$$

So we suppose that $e(X, Y) \geq n^2/8 + c'n^{3/2} + o(n^{3/2})$, with $c' > \sqrt{\ln 2}/2\sqrt{2}$ constant. Since Z is randomly chosen, we can estimate the the later quantities by

$$e(Z, Y) = \frac{e(X, Y)|Z|}{|X|}(1 + o(1)) \tag{3}$$

$$e(Z, X - Z) = e(X, X)\frac{|Z||X - Z|}{\frac{1}{2}|X|^2}(1 + o(1)). \tag{4}$$

Replacing by the known values, we get,

$$\begin{aligned}
e(Z,Y) &\leq \left(\frac{n^2}{8}+c'n^{3/2}\right)\frac{cn^{3/4}}{\frac{n}{2}+cn^{3/4}}(1+o(1)) \\
&\leq c\left(\frac{n^{7/4}}{4}+2c'n^{5/4}\right).
\end{aligned}$$

And for the other quantity, we need to estimate $e(X,X)$. The first moment method gives

$$e(X,X) \geq \frac{|X|^2}{2} - c''n^{3/2},$$

c" constant. Then, 4 gives

$$\begin{aligned}
e(Z,X-Z) &\geq |Z||X-Z|\left(\frac{1}{2}-\frac{2c''n^{3/2}}{|X|^2}\right)(1+o(1)) \\
&\geq \frac{cn^{7/4}}{2}\left(\frac{1}{2}-4c''n^{-1/2}\right)(1+o(1)).
\end{aligned}$$

Finally, we obtain

$$e(X',Y')-e(X,Y) = e(Z,X-Z)-e(Z,Y) \geq -2cc''n^{5/4}(1+o(1)),$$

and,

$$e(X',Y') > \frac{n^2}{8}+\frac{\sqrt{\ln 2}}{2\sqrt{2}}n^{3/2}+o(n^{3/2}).$$

which contradicts Corollary 1.

4 A lower bound for MAX-CUT

By using a greedy type algorithm we will obtain a lower bound for the maximum cut of a random graph. Actually the analysis of this particular algorithm provides us an upper bound for the value of $\mathrm{Bi}(G)$, with G distributed as $G_{n,1/2}$. The algorithm will be named *ALBI*.

4.1 The algorithm

Let $G=(V,E)$ be a random graph distributed as $G_{n,1/2}$ with n such that there exists an integer m verifying $n=2m^2$ (for the sake of simplicity, our algorithm will only consider graphs whose number of edges is twice a square). Let $V_1,\dots,V_m$ be m random sets partitionning V and such that for any $i\in\{1,\dots,m\}$, $|V_i|=2m$. In the following, $\Gamma(x)$ will denote the set of vertices linked to x by an edge, and for any $K,L\subset V$, $e(K,L)$ is the number of vertices between K and L.

The aim of this algorithm is to partition the vertices set of graph G into two sets X and Y such that the number of edges between these two sets is "small". The sets X and Y will be constructed iteratively by splitting the sets V_i ($1 \leq i \leq m$) at each step as to maximize the number of edges between X and Y. But keeping in mind the fact that in the end $|X| = |Y|$, at each step of the algorithm we store δ vertices apart in order to balance the sets X and Y. Z_k is the set created at the k^{th} step and containing those variables. We then need to compute the value of δ such that the sets are balanced with high probability.

The algorithm acts as follow. First we initialize all the variables. Next, we consider the set V_1 and split it randomly into the sets A_1 and B_1. Suppose that we have already constructed the sets A_{k-1} and B_{k-1} for some $k \in \{2, \ldots, m\}$. We then consider the set V_k. First, we chose randomly δ vertices of V_k and put them apart in a set Z_k. Then, for each remaining vertex of V_k we place it either in a set X_k if $|\Gamma(x) \cap A_{k-1}| - |\Gamma(x) \cap B_{k-1}| > 0$, or in a set Y_k if the contrary holds. When we are finished with the vertices of V_k, we place place the vertices of Z_k in such a manner that X_k and Y_k contains the same number of vertices. Finally we set $A_k = A_{k-1} \cup X_k$ and $B_k = B_{k-1} \cup Y_k$ and go on to the next k.

for k=1 **to** m

- $X_k := \emptyset$; $Y_k := \emptyset$; $Z_k := \emptyset$; $A_k := \emptyset$; $B_k := \emptyset$;

for k=1 **to** m

- chose x randomly in V;
- $X_1 := X_1 \cup x$;

$Y_1 := V_1 - X_1$; $A_1 := X_1$; $B_1 := Y_1$;
for k=2 **to** m

- chose randomly δ elements from V_k and place them in Z_k;
- **for** every element x of $V_k - Z_k$
  - **if** $|\Gamma(x) \cap A_{k-1}| - |\Gamma(x) \cap B_{k-1}| > 0$ **then** $X_k := X_k \cup \{x\}$
  - **if** $|\Gamma(x) \cap A_{k-1}| - |\Gamma(x) \cap B_{k-1}| < 0$ **then** $Y_k := Y_k \cup \{x\}$
  - **if** $|\Gamma(x) \cap A_{k-1}| - |\Gamma(x) \cap B_{k-1}| = 0$ **then** $X_k := x$ **or** $Y_k := \{x\}$ with probability $1/2$;
- X_k and Y_k are balanced with the elements of Z_k;
- $A_k := A_{k-1} \cup X_k$;
- $B_k := B_{k-1} \cup Y_k$;

4.2 Analysis of the algorithm

We need to determine a correct value for δ. At step k of the algorithm, we have to split the set V_k of size m between 2 subsets X_k and Y_k of equal size. So we want, with high probability, $|X_k| < m$ and $|Y_k| < m$. Since we split V_k into X_k and Y_k and $|V_k| = 2m - \delta$, $|X_k|$ and $|Y_k|$ are distributed as $\mathrm{B}(2m - \delta, 1/2)$. So, using a simple Chernoff bound,

$$\begin{aligned} \Pr\left[|X_k| > m\right] &= \Pr\left[|X_k| - m + \frac{\delta}{2} > \frac{\delta}{2}\right] \\ &\leq \exp\left(-\frac{\delta^2}{8m}\right) \end{aligned}$$

We conclude that we can chose $\delta = \sqrt{m}\log m$.

We want to estimate the number of edges $\widetilde{\mathrm{Bi}}(X, Y)$ between the sets X and Y produced by the algorithm. This quantity can be expressed by:

$$\widetilde{\mathrm{Bi}}(X, Y) = \sum_{k=2}^{m} \left[e(X_k, B_{k-1}) + e(Y_k, A_{k-1})\right] + \sum_{k=1}^{m} e(X_k, Y_k)$$

We set $C_1 = \sum_{k=2}^{m} \left[e(X_k, B_{k-1}) + e(Y_k, A_{k-1})\right]$ and $C_2 = \sum_{k=1}^{m} e(X_k, Y_k)$, then $\widetilde{\mathrm{Bi}}(X, Y) = C_1 + C_2$ and by linearity $\mathbb{E}\widetilde{\mathrm{Bi}}(X, Y) = \mathbb{E}C_1 + \mathbb{E}C_2$. C_1 is the number of edges between the vertices of V_k and the vertices previously considered for any k. C_2 is the number of edges between X_k and Y_k, for any k.

It is fairly easy to estimate C_2, since for every k, $e(X_k, Y_k)$ is distributed according to the binomial law $\mathrm{B}(m^2, 1/2)$. We then obtain,

$$\mathbb{E}C_2 = \frac{m^3}{2}.$$

We still need to estimate $\mathbb{E}C_1$. Considering the previous definition of C_1 we get

$$\mathbb{E}C_1 = \sum_{k=2}^{m} [\mathbb{E}e(X_k, B_{k-1}) + \mathbb{E}e(Y_k, A_{k-1})].$$

Let $k \in \{1 \ldots, m-1\}$ and $x \in V_{k+1}$. Clearly $e(x, A_k)$ and $e(x, B_k)$ follow the probability law $\mathrm{B}(km, 1/2)$ which can be approximated by a law of type $\mathrm{N}(\frac{km}{2}, \frac{km}{4})$. In the algorithm, x is placed in such a way that $e(x, B_k)$ or $e(x, A_k)$ are minimized. These 2 quantities are then distributed as the minimum of two normal laws. Then, it is quite well known that $\mathbb{E}e(x, B_k) = \mathbb{E}e(x, A_k) = \frac{km}{2} - \sqrt{\frac{km}{4\pi}}$. We must place $2m - \delta$ vertices according to the minimum of the 2 normal laws, and δ vertices to balance the subsets,

$$\begin{aligned}
\mathbb{E}C_1 &= (2m - \sqrt{m}\log m)\left[\sum_{k=1}^{m-1}\frac{km}{2} - \sum_{k=1}^{m-1}\frac{\sqrt{km}}{2\sqrt{\pi}}\right] + \sum_{k=1}^{m-1}\frac{km\delta}{2} \\
&= (2m - \sqrt{m}\log m)\left[\frac{m(m-1)}{2}\frac{m}{2} - \frac{m^2}{3\sqrt{\pi}}\right] + \frac{m(m-1)}{2}\frac{m\delta}{2} + o(m^3) \\
&= \frac{m^4}{2} - m^3\left(\frac{1}{2} + \frac{2}{3\sqrt{\pi}}\right) + o(m^3).
\end{aligned}$$

We deduce

$$\mathbb{E}\widetilde{\mathrm{Bi}}(X,Y) = \frac{m^4}{2} - \frac{2}{3\sqrt{2\pi}}m^3 + o(m^3),$$

as to say, with more conventionnal notations and setting $n = 2m^2$,

$$\begin{aligned}
\mathbb{E}\widetilde{\mathrm{Bi}}(X,Y) &= \frac{n^2}{8} - \frac{2}{6\sqrt{2\pi}}n^{3/2} + o(n^{3/2}) \\
&= \frac{n^2}{8} - 0,133n^{3/2} + o(n^{3/2}).
\end{aligned}$$

A straightforward use of the Chernoff bound (for more details, see [2]) shows that $\widetilde{\mathrm{Bi}}(X,Y)$ concentrates around its expectation. An easy application of the bound gives, for $1 \leq m \leq m$,

$$\Pr[|e(X_k,Y_k) - \mathbb{E}e(X_k,Y_k)| \geq m^2 t] \leq 2e^{-2m^2t^2},$$

and for $2 \leq k \leq m$,

$$\Pr[|e(X_k,B_{k-1}) - \mathbb{E}e(X_k,B_{k-1})| \geq 2km^2 t] \leq 2e^{-2m^2t^2},$$

and

$$\Pr[|e(Y_k,A_{k-1}) - \mathbb{E}e(Y_k,A_{k-1})| \geq 2km^2 t] \leq 2e^{-2m^2t^2}.$$

With correctly chosen t's we can then conclude

$$\Pr[|\widetilde{\mathrm{Bi}}(X,Y) - \mathbb{E}\widetilde{\mathrm{Bi}}(X,Y)| \geq m^{5/2}] = o(1).$$

This implies the theorem 3, since $m = \sqrt{n/2}$.

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